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The stability of a specific class of mechanical systems☆

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ABSTRACT

A special class of mechanical systems is considered, the linearized equations of which either belong to the class of time-varying systems, reducible to stationary systems using constructive Lyapunov transformations or to systems close to these. A method of decomposing of the matrices of a system, which differs from the traditional method, is proposed for investigating of the stability of motion. It is shown that the conclusions concerning the stability are more complete in the case of this decomposition of the system matrix. A number of problems on the stability of motion of various mechanical systems is considered as examples.

1. Linear time-varying systems of special form

Consider the first-order linear, time-varying system

$$\dot{x} = A(t)x \tag{1.1}$$

where x is the $n \times 1$ vector of the state of the system and $A(t)$ is an $n \times n$ matrix, the elements of which are continuously differentiable functions of the time t. Henceforth, $t \in [0, \infty)$ everywhere unless otherwise stated.

The case was considered in Ref. 1 when the matrix $A(t)$ satisfies the equation

$$\dot{A}(t) = CA(t) - A(t)C \tag{1.2}$$

where C is a constant $n \times n$ matrix. The transformation

$$x = \exp(Ct)y \tag{1.3}$$

reduces system (1.1) to the stationary form

$$\dot{y} = By, \quad B = A_0 - C, \quad A_0 = A(0)$$

and the fundamental matrix of system (1.1) is calculated using the formula

$$\Phi(t) = \exp(Ct)\exp(Bt)$$

A number of examples of systems of this class have been considered.[2,3]

It is well known that a system of equations of motion of holonomic mechanical systems that has been linearized in the neighbourhood of a certain preset motion can be represented in the form

$$N_1(t)\ddot{x} + N_2(t)\dot{x} + N_3(t)x = 0 \tag{1.4}$$

where x is an $n \times 1$ state vector and $N_i(t)(i = 1, 2, 3)$ is an $n \times n$ matrix with continuously differentiable elements.

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We will now consider a special class of mechanical systems that is of both theoretical and practical interest (see the examples below), the matrices $N_i(t)$ of which satisfy Eq. (1.2). The transformation (1.3) reduces system (1.4) to a stationary system of the form

$$L_0(y) = 0, \quad L_0(y) = M\ddot{y} + G\dot{y} + Ky$$

$$M = N_{10}, \quad G = 2N_{10}C + N_{20}, \quad K = N_{10}C^2 + N_{20}C + N_{30} \tag{1.5}$$

Here, $N_{i0} = N_i(0)$ $(i = 1, 2, 3)$.

The investigation of the solutions of the time-varying system (1.4) reduces to an investigation of the solutions of the stationary system (1.5) and an analysis of transformation (1.3).

The matrix $N_1(t)$ is symmetric in the majority of mechanical problems. The solutions of the matrix Eq. (1.2) for the matrices $N_i(t)$ have the form

$$N_i(t) = \exp(Ct)N_{i0}\exp(-Ct) \tag{1.6}$$

It can be shown that the matrix $N_1(t)$, represented in the form (1.6), satisfies the symmetry property if and only if the matrix C is skew-symmetric $C^T = -C$. The matrix $\exp(Ct)$ is then bounded and, consequently, the transformation (1.3) is a Lyapunov transformation.

The investigation of the stability of the time-varying system (1.4) therefore reduces to an analysis of the stability of the stationary system (1.5). The necessary and sufficient conditions for the stability of system (1.4) are determined by the roots of the characteristic equation of system (1.5)

$$\det[M\lambda^2 + G\lambda + K] = 0$$

In order to use the Kelvin-Chetayev theorems and their generalizations[4–8] to investigate the stability of system (1.5), it is necessary to consider the structure of the matrices of system (1.5).

It has already been pointed out that the matrix $N_1(t)$ is assumed to be symmetric. In the general case, the matrices G and K do not possess any symmetry property, even if the matrices $N_2(t)$ and $N_3(t)$ in the initial system (1.4) were symmetric or skew-symmetric.

Suppose $N_1(t) = E$ (E is a unit $n \times n$ matrix). We now calculate the conditions that the matrices of the initial system must satisfy in order for the matrices of the reduced system (1.5) to possess the following properties:

1) $G = G^T$, $K = K^T$ if and only if

$$N_{20}^T - N_{20} = 4C, \quad N_{30}^T - N_{30} = N_{20}C + CN_{20}^T$$

2) $G = G^T$, $K = -K^T$ if and only if

$$N_{20}^T - N_{20} = 4C, \quad N_{30}^T + N_{30} = 2C^2 + CN_{20} - N_{20}C$$

3) $G = -G^T$, $K = -K^T$ if and only if

$$N_{20}^T = -N_{20}, \quad N_{30}^T + N_{30} = -(2C^2 + CN_{20} + N_{20}C)$$

4) $G = -G^T$, $K = K^T$ if and only if

$$N_{20}^T = -N_{20}, \quad N_{30}^T - N_{30} = N_{20}C - CN_{20}$$

In particular, if $N_{20}C = CN_{20}$, then $N_2(t) = N_{20} = \text{const}$, $N_{30}^T = N_{30}$.

In this case, the reduced system retains the structure of the initial system, that is, a system containing time-varying potential forces and stationary gyroscopic forces reduces to the stationary system (1.5) with the same structure. The structure of the stationary system will also be the same when there are no gyroscopic forces ($N_2(t) \equiv 0$) in the initial time-varying system (1.4).

System (1.4), when

$$N_1(t) = E, \quad N_2(t) = N_{20}, \quad N_{20}C = CN_{20} \tag{1.7}$$

has a time-varying integral

$$V(t, x, \dot{x}) = \dot{x}^T\dot{x} + 2x^TC\dot{x} + x^T(N_3(t) + N_{20}C)x = \text{const}$$

It can be used as the Lyapunov function: the function V will be positive-definite if the matrix

$$K = C^2 + N_{20}C + N_{30} \tag{1.8}$$

is positive-definite and, by virtue of system (1.4), its total derivative is identically equal to zero, that is, the following theorem holds.

Theorem. *Suppose that, in system (1.4), the matrices $N_1(t)$ and $N_2(t)$ satisfy conditions (1.7) and that the symmetric matrix $N_3(t)$ satisfies condition (1.2). Then, system (1.4) is stable if the matrix K (1.8) is positive-definite.*

It is obvious that, in the case when the time-varying system is reducible to a stationary system using a Lyapunov transformation, an exhaustive analysis of the stability of the initial system can be carried out.

Example. Consider the system

$$\ddot{x} + A(t)x = 0; \quad A(t) = \left\| \begin{matrix} a + b\cos 2\omega t & -b\sin 2\omega t \\ -b\sin 2\omega t & a - b\cos 2\omega t \end{matrix} \right\|, \quad a, b = \text{const} \tag{1.9}$$

According to a well-known theorem,[9] this system has bounded solutions for sufficiently small $|b|$ if $a > 0$ and $m\omega \neq \sqrt{a}(m = 1, 2, \ldots)$. If, however, $a < 0$, it is impossible to draw any conclusions regarding the behaviour of this system on the basis of known theorems.

At the same time, it can be shown that the matrix $A(t)$ satisfies Eq. (1.2) when $C = \left\| \begin{matrix} 0 & \omega \\ -\omega & 0 \end{matrix} \right\|$. The Lyapunov transformation (1.3) reduces system (1.9) to the stationary system

$$\ddot{y} + 2C\dot{y} + (C^2 + A_0)y = 0 \tag{1.10}$$

the characteristic equation of which can be represented in the form

$$\lambda^4 + 2p\lambda^2 + q = 0; \quad p = a + \omega^2, \quad q = (a - \omega^2)^2 - b^2$$

The necessary and sufficient conditions for system (1.10) and, consequently, system (1.9) to be stable have the form

$$a + \omega^2 > 0, \quad -4a\omega^2 < b^2 < (a - \omega^2)^2 \tag{1.11}$$

Conditions (1.11) show that the system is stable when $a > 0$ in the case of the constraints imposed on the parameter b. If $\omega^2 = a$, the system is unstable for any values of the parameter $b \neq 0$.

If $a < 0$, the degree if instability of system (1.10) will be even and, if $\omega^2 > a + b$, gyroscopic stabilization of the system is ensured for a sufficiently high frequency ω and values of the parameter b lying in the bounded domain (1.11), the width of which $(a + \omega^2)^2$ tends to zero when $\omega^2 \to |a|$.

The investigation of the stability of linear time-varying systems of the general form of (1.1) is a difficult problem that has not yet been completely solved. The traditional method[9–12] for investigating of the stability of such systems consists of representing the system matrix in the form of two terms: $A(t) = A^0 + A_1(t)$, the first of which is constant and the second of which is small in a certain sense. Conclusions concerning the stability of the initial system can be drawn from the stability properties of the stationary system with the matrix A^0 when specific conditions are satisfied.[9–12]

The proposed approach consists of the following. We assume that a different partitioning of the coefficient matrix of system (1.1): $A(t) = \tilde{A}^0(t) + \tilde{A}_1(t)$ is possible such that the matrix $\tilde{A}_1(t)$ is small and the system

$$\dot{x} = \tilde{A}^0(t)x \tag{1.12}$$

is reducible to a stationary system by means of a constructive Lyapunov transformation

$$x = T(t)y \tag{1.13}$$

In particular, if system (1.12) is integrable in closed form and its fundamental matrix $\Phi(t, 0)$ and the matrix $\Phi^{-1}(t, 0)$ are bounded, then $T(t) = \Phi(t, 0)$. The transformation (1.13) then reduces system (1.1) to the form

$$\dot{y} = (B^0 + B_1(t))y; \quad B^0 = \text{const}, \quad B_1(t) = T^{-1}(t)A_1(t)T(t) \tag{1.14}$$

where the matrix $B_1(t)$ is small.

As indicated above, well-known theorems on the stability of linear systems with an almost constant matrix[9–12] can be used to analyze the stability of system (1.14). In this method of decomposition of the matrix of the initial system, the information on a dynamical object contained in the matrix $A(t)$ is used more fully which naturally leads to more accurate conclusions regarding stability.

We will assume that the matrices $N_i(t)$ in system (1.4) can be represented in the form

$$N_i(t) = N_i^0(t) + \varepsilon R_i(t, \varepsilon) \tag{1.15}$$

where the matrices $N_i^0(t)$ satisfy Eq. (1.2), $\varepsilon > 0$ is a small parameter and the elements of the matrices $R_i(t, \varepsilon)$ are continuous bounded functions of t and analytic functions of the parameter ε in the neighbourhood of $\varepsilon = 0$.

In this case, transformation (1.3) reduces system (1.4) to the form

$$\begin{aligned} &L^0 y = -\varepsilon L^1 y \\ &L^1 y = K_1\ddot{y} + (2K_1C + K_2)\dot{y} + (K_1C^2 + K_2C + K_3)y, \\ &K_i(t) = \exp(-Ct)N_i(t)\exp(Ct) \end{aligned} \tag{1.16}$$

where $K_i(t)$ are bounded functions and the operator L^0y is defined by the second formula of (1.5).

By changing from system (1.4) to a system of first order equations in Cauchy form, it can be shown that, in view of the boundedness of the matrices $K_i(t)$, assertions hold which are analogous to the theorems on the stability of first-order systems with an almost constant matrix.[9–12]

2. The problem of the parametric oscillations of a bearing housing and a shaft[13]

A shaft of length l and mass per unit length m has an eccentricity e and is located in a housing mounted on three elastic shock absorbers. When the shaft rotates with an angular velocity ω, the housing oscillates. The position of the axis of rotation of the shaft relative to a fixed system of coordinates is characterized by small angles of deviations from the vertical α and β.

The linearized system of equations of motion has the form[13]

$$A(t)\ddot{x} + \varepsilon\omega G(t)\dot{x} + Kx = 0; \quad x = \mathrm{col}(x_1, x_2), \quad x_1 = \alpha, \quad x_2 = \beta \tag{2.1}$$

Here,

$$A(t) = \begin{Vmatrix} 1+\varepsilon\sin^2\omega t & \frac{1}{2}\varepsilon\sin 2\omega t \\ \frac{1}{2}\varepsilon\sin 2\omega t & 1+\varepsilon\cos^2\omega t \end{Vmatrix}, \quad G(t) = \begin{Vmatrix} \sin 2\omega t & -2\sin^2\omega t \\ 2\cos^2\omega t & -\sin 2\omega t \end{Vmatrix}, \quad K = \nu^2 E_2$$

ν is the frequency of natural oscillations of the housing and $\varepsilon > 0$ is a certain parameter.

It can be shown that the matrices $A(t)$ and $G(t)$ satisfy an equation of the form of (1.2) with the matrix $C = \begin{Vmatrix} 0 & \omega \\ -\omega & 0 \end{Vmatrix}$ and $KC = CK$. In this case, a transformation of the form of (1.3), which is a Lyapunov transformation, reduces system (2.1) to the stationary system

$$A(0)\ddot{y} + (2A(0)C + \varepsilon\omega G(0))\dot{y} + (K + A(0)C^2 + \varepsilon\omega G(0)C)y = 0 \tag{2.2}$$

The characteristic equation of this system

$$\det\begin{Vmatrix} \lambda^2+\kappa_1 & 2\omega\lambda \\ -2\omega\lambda & (1+\varepsilon)\lambda^2+\kappa_2 \end{Vmatrix} = 0; \quad \kappa_1 = \nu^2-\omega^2, \quad \kappa_2 = \nu^2-\omega^2(1-\varepsilon)$$

can be represented in the form

$$a_0\lambda^4 + a_1\lambda^2 + a_2 = 0$$

$$a_0 = 1+\varepsilon, \quad a_1 = \nu^2(2+\varepsilon)+2\omega^2, \quad a_2 = (\nu^2-\omega^2)[\nu^2-\omega^2(1-\varepsilon)]$$

The conditions for system (2.2) to be stable have the form

$$a_2 > 0, \ \Delta = 4\varepsilon^2\chi^2 + 4(4+2\varepsilon-\varepsilon^2)\chi + \varepsilon^2 > 0; \ \chi = \omega^2/\nu^2 \tag{2.3}$$

It can be shown that $\Delta > 0$ for all $\chi > 0$. The sole essential stability condition is therefore the first equality of (2.3) from which the exact boundaries of the domain of instability are determined:

when $0 < \varepsilon < 1$, the system is unstable if $\nu < \omega < \nu/\sqrt{1-\varepsilon}$

when $\varepsilon < 1$, the system is unstable if $\omega > \nu$.

The domain of instability of system (2.1) was constructed[13] using of the theory of parametric resonance, with the assumption that ε is a small parameter.

3. The stability of the periodic motion of a heavy rotor

We will now consider a heavy, symmetric rotor mounted with a certain eccentricity e half way along a horizontally positioned elastic shaft. We shall assume that the rotor executes plane-parallel motion. Suppose l is the distance from the point O, where the rotor is attached to the shaft to the centre of mass of the rotor G. Assuming that there are no internal and external friction forces, the system of equations of motion of the rotor in the fixed system of coordinates $O\xi_1\xi_2$ has the form[12]

$$X_1'' + X_1 - \varepsilon\cos\varphi = 0, \quad X_2'' + X_2 - \varepsilon\sin\varphi = 0$$

$$\varphi'' + \varepsilon(X_1\sin\varphi - X_2\cos\varphi) = \varepsilon\delta\cos\varphi \tag{3.1}$$

Here,

$$\tau = \omega t, \quad \omega = \sqrt{\frac{c}{m}}, \quad X_1 = \frac{\xi_1}{\rho}, \ X_2 = \frac{\xi_2}{\rho} - \delta, \ \delta = \frac{g}{\omega^2\rho}, \ \varepsilon = \frac{l}{\rho}$$

c is the flexural stiffness of the shaft, m and ρ are the mass and radius of inertia of the rotor, ξ_j are the coordinates of the centre of mass of the rotor in the system of coordinates $O\xi_1\xi_2$ and φ is the angle between the axis $O\xi_1$ and the vector OG. A prime denotes a derivative with respect to τ.

The system of Eq. (3.1) admits of the particular solution[12]

$$X_{10} = -\frac{4}{3}\varepsilon\cos\frac{\tau}{2} + \delta\sin\tau, \quad X_{20} = -\frac{4}{3}\varepsilon\sin\frac{\tau}{2} - \delta\cos\tau, \quad \varphi_0 = \pi + \frac{\tau}{2} \tag{3.2}$$

The system of Eq. (3.1), linearized in the neighbourhood of this particular solution, has the form

$$x'' + A(\tau)x = 0; \; x = \mathrm{col}(x_1, x_2, x_3), \; x_1 = X_1 - X_{10}, \; x_2 = X_2 - X_{20}, \; x_3 = \varphi - \varphi_0 \tag{3.3}$$

An analysis of the stability of system (3.3) was carried out[12] by representating the matrix $A(\tau)$ in the form of a sum of constant matrices and a time-varying matrix with periodic elements

$$A(\tau) = A^{(0)} + \varepsilon A^{(1)}(\tau) + \varepsilon^2 A^{(2)}$$

$$A^{(0)} = \mathrm{diag}(1, 1, 0), \; A^{(1)}(\tau) = \begin{Vmatrix} 0 & 0 & s \\ 0 & 0 & -c \\ s & -c & -2\delta s \end{Vmatrix}, \; A^{(2)} = \mathrm{diag}\left(0, 0, \frac{4}{3}\right), \; s = \sin\frac{\tau}{2}, \; c = \cos\frac{\tau}{2} \tag{3.4}$$

The characteristic equation of the stationary unperturbed system ($\varepsilon = 0$) has the double roots $0, \pm i$.

A method has been proposed for calculating the characteristic exponents of systems of the form (3.3), (3.4), on the basis of which formulae were obtained for the characteristic exponents in the above-mentioned problem under the assumption that ε was of small magnitude.[12]

We will now consider another method of partitioning the matrix $A(t)$, in which we assume that the parameter δ is small and that the parameter ε is arbitrary

$$A(\tau) = A_1(\tau, \varepsilon) + \delta A_2(\tau, \varepsilon)$$

$$A_1(\tau, \varepsilon) = \begin{Vmatrix} 1 & 0 & \varepsilon s \\ 0 & 1 & -\varepsilon c \\ \varepsilon s & -\varepsilon c & \frac{4}{3}\varepsilon^2 \end{Vmatrix}, \quad A_2(\tau, \varepsilon) = \mathrm{diag}(0, 0, -2\varepsilon s) \tag{3.5}$$

It was found that the matrix $A_1(\tau, \varepsilon)$ satisfies Eq. (1.2). The elements of the matrix C are zeros apart from $c_{12} = -c_{21} = -1/2$. Using transformation (1.3), system (3.3) is reduced to the form

$$z'' + 2Cz' + [C^2 + A_1(0, \varepsilon) + \delta A_2(\tau, \varepsilon)]z = 0 \tag{3.6}$$

We now represent Eq. (3.6) in the form

$$y' = [W(\varepsilon) + \delta R(\tau, \varepsilon)]y, \quad y = \mathrm{col}(z, z')$$

$$W(\varepsilon) = \begin{Vmatrix} O_3 & E_3 \\ W_{21} & W_{22} \end{Vmatrix}, \quad R(\tau, \varepsilon) = \begin{Vmatrix} O_3 & O_3 \\ -A_2(\tau, \varepsilon) & O_3 \end{Vmatrix}, \quad W_{21}(\varepsilon) = \begin{Vmatrix} -\frac{3}{4} & 0 & 0 \\ 0 & -\frac{3}{4} & \varepsilon \\ 0 & \varepsilon & -\frac{4}{3}\varepsilon^2 \end{Vmatrix}$$

$$W_{22} = -2C \tag{3.7}$$

where O_3 and E_3 are null and unit 3×3 matrices.

The analysis of the stability of system (3.3) using representation (3.4) is based on an examination of the eigenvalues of the constant matrix A_0, which does not contain the parameter ε. Application of the partitioning (3.5) leads to an investigation of system (3.7) in which the constant matrix W contains the parameter ε. When $\delta = 0$, the exact solution of system (3.7) $y(t) = \exp(W(\varepsilon)t)y(0)$ can be written out.

The eigenvalues α^0 of the matrix $W(\varepsilon)$ have the form

$$\alpha^0_{1,2} = 0, \quad \alpha^0_{3,4} = \pm i\omega_1, \; \alpha^0_{5,6} = \pm i\omega_2, \; \omega^2_{1,2} = \frac{5}{4} + \frac{2}{3}\varepsilon^2 \pm \sqrt{1 - \frac{2}{3}\varepsilon^2 + \frac{4}{9}\varepsilon^4} \tag{3.8}$$

Assuming that the magnitude of δ is small, the characteristic exponents

$$\alpha = \alpha^0(\varepsilon) + \delta\alpha^{(1)}(\varepsilon) + \delta^2\alpha^{(2)}(\varepsilon) + O(\delta^{2+\mu}) \; (\mu > 0)$$

of system (3.7) can be calculated using a well-known method.[12]

It can be shown that the correction to the characteristic exponent $\alpha^0_{1,2}=0$, with an accuracy of up to terms in $O(\delta^2)$ inclusive, is equal to zero and that the characteristic exponents $\alpha_{3,4}$, $\alpha_{5,6}$ do not contain the parameter δ in the first power $\alpha_k^{(1)}(\varepsilon)=0$, $k=3, 4, 5, 6$.

The corrections $\alpha_k^{(2)}(\varepsilon)$ $k=3, 4, 5, 6$ have the form

$$\alpha_k^{(2)} = if_k(\varepsilon),\ f_k(\varepsilon) = \frac{\varepsilon^4}{2\omega_j\nu_j}\left[\frac{\Delta_3(\Omega_j)}{D(\Omega_j)} - \frac{\Delta_3(\Omega_j')}{\Delta(\Omega_j')}\right],\ k = 3,\ j = 1;\ k = 5,\ j = 2$$

$$\Omega_j = \omega_j + \frac{1}{2},\quad \Omega_j' = \omega_j - \frac{1}{2},\quad \Delta_3(\omega) = \kappa^2(\omega) - \omega^2,\quad D(\omega) = \Delta(\omega)\Delta_3(\omega) - \varepsilon^2\kappa(\omega)$$

$$\kappa(\omega) = \omega^2 - \frac{3}{4},\quad \Delta(\omega) = \omega^2 - \frac{4}{3}\varepsilon^2,\quad \nu_j = \Delta^2(\omega_j) + \varepsilon^2 + \frac{3}{4}\frac{\Delta^2(\omega_j)}{\kappa^2(\omega_j)}$$

$$\alpha_4^{(2)} = -\alpha_3^{(2)},\quad \alpha_6^{(2)} = -\alpha_5^{(2)} \tag{3.9}$$

Formulae (3.9) hold for any values of the parameter ε. For small ε, we obtain from formulae (3.9)

$$\alpha_{1,2} = 0,\quad \alpha_{3,4} = \pm\frac{3}{2}i \pm \frac{1}{9}\varepsilon^2 i,\quad \alpha_{5,6} = \pm\frac{i}{2}(1 + 2\varepsilon^2 + 4\delta^2 - 32.6\varepsilon^2\delta^2)$$

with an accuracy up to ε^3, which corresponds to the approximate formulae found earlier,[12] where the parameter δ was assumed to be finite and the parameter ε was assumed to be small.

4. A gyroscopic servosystem[14]

A gyroscopic servosystem has been considered with equations which, when there are no stochastic effects, can be represented in the form[14]

$$\ddot{\alpha} + k_0\Omega\dot{\beta} = -u(t)\sin\Omega t,\quad \ddot{\beta} - k_0\Omega\dot{\alpha} = u(t)\cos\Omega t \tag{4.1}$$

Here, α and β are the angles defining the position of the axis of the gyroscope, Ω is the angular velocity of rotation of the rotor of the gyroscope, $k_0\Omega$ ($k_0=\text{const}>0$) is the relative angular momentum and $u(t)$ is the control moment, the expression for which is taken in the form[14]

$$u(t) = 2k_2(\dot{\beta}\cos\Omega t - \dot{\alpha}\sin\Omega t) + 2k_1(\alpha\cos\Omega t + \beta\sin\Omega t)$$

The constant coefficients k_1 and k_2 remain to be chosen.

This system belongs to the type of two-channel systems with modulation and a single inertialess alternating current channel.[14]

System (4.1) can be represented in the form of (1.4), where $N_1(t)\equiv E_2$ and

$$N_2 = \begin{Vmatrix} -k_2(1-\cos 2\Omega t) & k_0\Omega + k_2\sin 2\Omega t \\ -k_0\Omega + k_2\sin 2\Omega t & -k_2(1+\cos 2\Omega t) \end{Vmatrix},\quad N_3 = k_1\Omega\begin{Vmatrix} \sin 2\Omega t & 1-\cos 2\Omega t \\ -1-\cos 2\Omega t & -\sin 2\Omega t \end{Vmatrix}$$

The matrices $N_2(t)$ and $N_3(t)$ satisfy Eq. (1.2) when $C = \begin{Vmatrix} 0 & \Omega \\ -\Omega & 0 \end{Vmatrix}$. Transformation (1.3) reduces system (4.1) to a stationary system of the form (1.5)

$$G = \begin{Vmatrix} 0 & \Omega(k_0-2) \\ -\Omega(k_0-2) & -2k_2 \end{Vmatrix},\quad K = \begin{Vmatrix} \Omega^2(k_0-1) & 0 \\ -2\Omega(k_1+k_2) & \Omega^2(k_0-1) \end{Vmatrix} \tag{4.2}$$

The characteristic equation of system (4.2) has the form

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

$$a_1 = -2k_2,\quad a_2 = \Omega^2(k_0^2 - 2k_0 + 2),\quad a_3 = 2\Omega^2[(k_0-2)k_1 - k_2],\quad a_4 = \Omega^4(k_0-1)^2$$

According to the Hurwitz criterion, the necessary and sufficient conditions for the asymptotic stability of system (4.2) when $k_0\neq 1, 2$ can be represented in the form

$$k_2 < 0,\quad 0 < k_1 < -k_0k_2 \tag{4.3}$$

System (4.2) is simply stable when $k_0=1$ and $k_0=2$ and the inequalities (4.3) are satisfied. Conditions (4.3) can be satisfied by a corresponding choice of the control coefficients k_1 and k_2.

In this case, the transformation (1.3) is a Lyapunov transformation and the stability conditions that have been obtained are therefore the stability conditions of the initial system (4.1).

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